# MATHEMATICAL MODEL FOR VERTICAL DISTRIBUTION OF VELOCITY IN CHANNELS. A CASE STUDY 

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#### Abstract

The present paper, based on the common equation of hydraulics and on a selection of quantitative specifications from literature dealing with this subject, consists of a double-parabolic theoretical law referring to the vertical distribution of velocities in open channels. Then, based on this distribution law, algebraic expressions for the relative quota corresponding to the average velocity as well as for the assessment of the Coriolis and Boussinesq coefficients are deduced. For a concrete example of calculation, the algebraic relations deduced are verified using numerical methods.


Key words: Open channel, vertical distribution of velocities, double-parabolic law, Coriolis coeficient, Boussinesq coefficient.

Using the common equation of hydraulic calculations the average velocity in the active cross-section of a free surface stream in continuous flow can be calculated and the average velocity can also be estimated on one of the stream's vertical.

The Coriolis coefficient $\alpha$ and Boussinesq coefficient $\beta$ interfere in the energy and impulse equations; they can be assessed if the velocity distribution on one vertical of the stream and/or on its entire active cross-sectional area are known.

The velocity distribution can also facilitate the evaluation of other values, for example the drag of a solid object, local resistance induced by a singularity, and longitudinal and/or transverse hydrodynamic dispersion coefficients.

Based on the common hydraulics equations and some quantitative specifications from the specialized literature, a double-parabolic theoretical law on the vertical distribution of velocities in open channels is defined.

Later, starting from this distribution law, algebraic expression are deduced for the vertical stream and for the relative level corresponding to the average velocity, as well as for the evaluation of the $\alpha$ and $\beta$ coefficients. For an example of a concrete calculation, all the newly deduced algebraic relations are verified using numerical methods.

## MATERIAL AND METHOD

Referring to velocity distribution in a channel with a free surface, on the vertical with a water
depth of h, some authors, surmise that due to friction with the atmospheric air the velocity at the top water-surface stream, vs, is inferior to the maximum velocity of flow, vmax, registered at the
zmax level - at $(1 / 6 \cdots 1 / 5) \cdot h$ below the water surface - thus (Bartha I., 2004):

$$
\tilde{z}_{\max }=z_{\max } / h \in[4 / 5,5 / 6], v_{\max }=\varphi \cdot v_{s}
$$

with $\varphi \cong 1.29$
where zmax represents the relative quota corresponding to velocity vmax.

The velocity at the invert channel, vb, will be approximated using the friction velocity $\mathrm{v}^{*}$, (Mateescu C., 1993):

$$
\begin{equation*}
v_{f} \cong v_{*}=\sqrt{g \cdot R \cdot I_{h}} \tag{2}
\end{equation*}
$$

where g represents acceleration due to gravity, R is the hydraulic radius, and Ih the hydraulic slope.

The average velocity on a vertical stream of depth h can be estimated with the Chezy formula:

$$
\begin{equation*}
V=C \sqrt{R \cdot I_{h}} \cong(1 / n) \cdot R^{1 / 6} \sqrt{R \cdot I_{h}} \tag{3}
\end{equation*}
$$

where: C is Chezy's coefficient, and $n$ is the coefficient of roughness, referred to as Manning's roughness coefficient.

Later on we will use the relative velocities at the bottom, $\tilde{v}_{f}$, and at the surface, $\tilde{v}_{s}$, and the maximum relative velocity. $\tilde{v}_{\text {max }}$, defined by the following relations:

[^0]$\tilde{v}_{f}=v_{f} / V ; \quad \tilde{v}_{s}=v_{s} / V ; \tilde{v}_{\max }=v_{\max } / V$,

From relations (4) and (1) it results that: $\tilde{v}_{\max }=\varphi \cdot \tilde{v}_{s}$, with $\varphi \cong 1.29$
while from relations (4), (2) and (3) and (1), the result is:

$$
\begin{equation*}
\tilde{v}_{f} \cong v_{*} / V=n \sqrt{g} \cdot R^{-1 / 6} \tag{6}
\end{equation*}
$$

## RESULTS AND DISCUSSIONS

Based on relations (1), (5) and (6) a doubleparabolic theoretical law on the vertical distribution of velocities in channels was determined.

As practical applications of this distribution law for that particular vertical, algebraic expressions were inferred for the relative level corresponding to the average velocity $\tilde{z}_{\text {med }}$ as well as for the evaluation of coefficients $\alpha$ and $\beta$.

In the end, all the newly deduced algebraic relations were verified using numerical methods, within an appropriate case study.

## The double-parabolic law on the vertical distribution of velocities in channels

The relative velocity profile $\tilde{v}=\mathrm{f}(\tilde{z})$ was estimated on sections using the following second degree polynomial functions (7):
$\tilde{v}=\mathrm{f}(\tilde{z})=\left\{\begin{array}{l}a_{1} \cdot(\tilde{z})^{2}+b_{1} \cdot \tilde{z}+c_{1}, \text { for } 0 \leq \tilde{z} \leq \tilde{z}_{\text {max }} \\ a_{2} \cdot(\tilde{z})^{2}+b_{2} \cdot \tilde{z}+c_{2}, \text { for } \tilde{z}_{\text {max }} \leq \tilde{z} \leq 1\end{array}\right.$
Using characteristic values (1), (5) and (6), the points imposed on function (7) graph were deduced:

$$
\begin{equation*}
(\tilde{z}, \tilde{v}) \in\left\{\left(0, \tilde{v}_{f}\right),\left(1, \tilde{v}_{s}\right),\left(\tilde{z}_{\max }, \tilde{v}_{\max }\right)\right\} \tag{8}
\end{equation*}
$$

Constraining the maximum point to check both branches of function (7), $\tilde{v}_{\max }=\mathrm{f}\left(\tilde{z}_{\text {max }}\right)$, the equations in (7) can also be presented as follows (9):

$$
\tilde{v}=\mathrm{f}(\tilde{z})=\left\{\begin{array}{l}
a_{1} \cdot\left(\tilde{z}-\tilde{z}_{\max }\right)^{2}+\tilde{v}_{\max }, \text { for } 0 \leq \tilde{z} \leq \tilde{z}_{\max } \\
a_{2} \cdot\left(\tilde{z}-\tilde{z}_{\max }\right)^{2}+\tilde{v}_{\max }, \text { for } \tilde{z}_{\max } \leq \tilde{z} \leq 1
\end{array}\right.
$$

Equations

$$
\tilde{v}_{f}=\mathrm{f}(0) \Leftrightarrow a_{1} \cdot\left(\tilde{z}_{\max }\right)^{2}+\tilde{v}_{\max }=\tilde{v}_{f}
$$

and

$$
\tilde{v}_{s}=\mathrm{f}(1) \Leftrightarrow a_{1} \cdot\left(1-\tilde{z}_{\max }\right)^{2}+\tilde{v}_{\max }=\tilde{v}_{s}
$$

resulted in the following solutions for coefficients $a_{1}$ and $a_{2}$ :

$$
\begin{equation*}
a_{1}=-\frac{\tilde{v}_{\max }-\tilde{v}_{f}}{\left(\tilde{z}_{\max }\right)^{2}}, a_{2}=-\frac{\tilde{v}_{\max }-\tilde{v}_{s}}{\left(1-\tilde{z}_{\max }\right)^{2}} \tag{10}
\end{equation*}
$$

Introducing expression (10) in relation (9) resulted in the subsequent analytical expression, in relative coordinates for the velocity profile (11):
$\tilde{v}=\mathrm{f}(\tilde{z})=\left\{\begin{array}{l}\tilde{v}_{\text {max }}-\frac{\tilde{v}_{\text {max }}-\tilde{v}_{f}}{\left(\tilde{z}_{\text {max }}\right)^{2}} \cdot\left(\tilde{z}-\tilde{z}_{\text {max }}\right)^{2}, \text { for } 0 \leq \tilde{z} \leq \tilde{z}_{\text {max }} \\ \tilde{v}_{\text {max }}-\frac{\tilde{v}_{\text {max }}-\tilde{v}_{s}}{\left(1-\tilde{z}_{\text {max }}\right)^{2}} \cdot\left(\tilde{z}-\tilde{z}_{\text {max }}\right)^{2}, \text { for } \tilde{z}_{\text {max }} \leq \tilde{z} \leq 1\end{array}\right.$

Function (11) must comply with the next integral condition:
$\int_{0}^{1} \mathrm{f}(\tilde{z}) \cdot \mathrm{d} \tilde{z}=1$
Bearing in mind the definition on the sections of function $\tilde{v}=\mathrm{f}(\tilde{z})$, the definite integral, from the left-hand term of the above stated relation, can be evaluated as the sum of three definite integrals:

$$
\begin{align*}
& \int_{0}^{1} \mathrm{f}(\tilde{z}) \cdot \mathrm{d} \tilde{z}=-\frac{\tilde{v}_{\max }-\tilde{v}_{f}}{\left(\tilde{z}_{\max }\right)^{2}} \cdot \int_{0}^{\tilde{z}_{\max }}\left(\tilde{z}-\tilde{z}_{\max }\right)^{2} \cdot \mathrm{~d} \tilde{z}  \tag{13}\\
& -\frac{\tilde{v}_{\max }-\tilde{v}_{s}}{\left(1-\tilde{z}_{\max }\right)^{2}} \cdot \int_{\tilde{z}_{\max }}^{1}\left(\tilde{z}-\tilde{z}_{\max }\right)^{2} \cdot \mathrm{~d} \tilde{z}+\tilde{v}_{\max } \cdot \int_{0}^{1} \mathrm{~d} \tilde{z}
\end{align*}
$$

Putting into practice the integration rules of polynomial functions, successively results in:

$$
\begin{align*}
& \int_{0}^{1} \mathrm{f}(\tilde{z}) \cdot \mathrm{d} \tilde{z}=-\frac{\tilde{v}_{\max }-\tilde{v}_{f}}{\left(\tilde{z}_{\max }\right)^{2}} \cdot \frac{1}{3} \cdot\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{3}\right]_{0}^{\tilde{z}_{\max }}  \tag{14}\\
& -\frac{\tilde{v}_{\max }-\tilde{v}_{s}}{\left(1-\tilde{z}_{\max }\right)^{2}} \cdot \frac{1}{3} \cdot\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{3}\right]_{\tilde{z}_{\max }}^{1}+\tilde{v}_{\max } \cdot[\tilde{z}]_{0}^{1} \\
& \int_{0}^{1} \mathrm{f}(\tilde{z}) \cdot \mathrm{d} \tilde{z}=\left[2 \cdot \tilde{v}_{\max }+\tilde{v}_{s}-\left(\tilde{v}_{s}-\tilde{v}_{f}\right) \cdot \tilde{z}_{\max }\right] / 3
\end{align*}
$$

Conditioning the integral (14) to check relation (12) resulted in:

$$
\begin{equation*}
\left[2 \cdot \tilde{v}_{\max }+\tilde{v}_{s}-\left(\tilde{v}_{s}-\tilde{v}_{f}\right) \cdot \tilde{z}_{\max }\right] / 3=1 \tag{15}
\end{equation*}
$$

Solving the system formed by equations (5) and (15), the subsequent relations for the relative velocities $\tilde{v}_{s}$ and $\tilde{v}_{\max }$ resulted $\operatorname{in}(16)$ :

$$
\tilde{v}_{s}=\frac{3-\tilde{v}_{f} \cdot \tilde{z}_{\max }}{1+2 \cdot \varphi-\tilde{z}_{\max }}, \tilde{v}_{\max }=\varphi \cdot \frac{3-\tilde{v}_{f} \cdot \tilde{z}_{\max }}{1+2 \cdot \varphi-\tilde{z}_{\max }}
$$

## The algebraic expression for the relative quota $\tilde{z}_{\text {med }}$

The relative average velocity is equal to 1 and corresponds to the first branch of the function $\tilde{v}=\mathrm{f}(\tilde{z})$ given by relation (11) for $\tilde{z}=\tilde{z}_{\text {med }}$, where $\tilde{z}_{\text {med }}$ is the relative quota corresponding to the average velocity:
$\tilde{v}_{\text {max }}-\left(\tilde{v}_{\text {max }}-\tilde{v}_{f}\right) \cdot\left[\left(\tilde{z}_{\text {med }}-\tilde{z}_{\text {max }}\right) / \tilde{z}_{\text {max }}\right]^{2}=1$
The solution of the above-mentioned second degree equation, for the sign alternance that respects the condition $\tilde{z}_{\text {med }}<\tilde{z}_{\text {max }}$, is the following:

$$
\begin{equation*}
\tilde{z}_{\operatorname{med}}=\tilde{z}_{\max } \cdot\left(1-\tilde{v}_{f}\right) /\left(\tilde{v}_{\max }-\tilde{v}_{f}\right) \tag{17}
\end{equation*}
$$

## Algebraic expressions for Boussinesq coefficient, $\beta$, and Coriolis coefficient, $\alpha$

Boussinesq and Coriolis coefficients, respectively, are provided by the following integrals [2]:
$\beta=\int_{0}^{1} \mathrm{f}^{2}(\tilde{z}) \cdot d \tilde{z}$
and
$\alpha=\int_{0}^{1} \mathrm{f}^{3}(\tilde{z}) \cdot d \tilde{z}$
Introducing into relation (18) equation (9) for the function $\mathrm{f}(\tilde{z})$, results in the subsequent sum of definite integrals on disjoint intervals:

$$
\begin{aligned}
& \beta=\int_{0}^{1} \mathrm{f}^{2}(\tilde{z}) \cdot d \tilde{z}= \\
& \int_{0}^{\tilde{z}_{\max }}\left[\tilde{v}_{\max }+a_{1} \cdot\left(\tilde{z}-\tilde{z}_{\max }\right)^{2}\right]^{2} \cdot \mathrm{~d} \tilde{z} \\
& +\int_{\tilde{z}_{\max }}^{1}\left[\tilde{v}_{\max }+a_{2} \cdot\left(\tilde{z}-\tilde{z}_{\max }\right)^{2}\right]^{2} \cdot \mathrm{~d} \tilde{z}
\end{aligned}
$$

Raising to the second power the binomial expressions of each of the above-mentioned integrals, and disregarding the constant factors below the integration symbol, results in the following sum of integrals from the power functions:

$$
\begin{aligned}
& \beta=\left(\tilde{v}_{\max }\right)^{2} \cdot \int_{0}^{\tilde{z}_{\max }} \mathrm{d} \tilde{z}+2 \cdot \tilde{v}_{\max } \cdot a_{1} \cdot \\
& \int_{0}^{\tilde{z}_{\max }}\left(\tilde{z}-\tilde{z}_{\max }\right)^{2} \cdot \mathrm{~d} \tilde{z}+\left(a_{1}\right)^{2} \cdot \int_{0}^{\tilde{z}_{\max }}\left(\tilde{z}-\tilde{z}_{\max }\right)^{4} \cdot \mathrm{~d} \tilde{z}+ \\
& \left(\tilde{v}_{\max }\right)^{2} \cdot \int_{\tilde{z}_{\max }^{1}}^{1} \mathrm{~d} \tilde{z}+2 \cdot \tilde{v}_{\max } \cdot a_{2} \cdot \\
& \int_{\tilde{z}_{\max }}^{1}\left(\tilde{z}-\tilde{z}_{\max }\right)^{2} \cdot \mathrm{~d} \tilde{z}+\left(a_{2}\right)^{2} \cdot \int_{\tilde{z}_{\max }}^{1}\left(\tilde{z}-\tilde{z}_{\max }\right)^{4} \cdot \mathrm{~d} \tilde{z}
\end{aligned}
$$

Next, after uniting the disjointed intervals and performing the integration operations, results in the following:

$$
\begin{aligned}
& \beta=\left(\tilde{v}_{\max }\right)^{2} \cdot \int_{0}^{1} \mathrm{~d} \tilde{z}+\frac{2}{3} \cdot \tilde{v}_{\max } \cdot a_{1} \\
& \cdot\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{3}\right]_{0}^{\tilde{z}_{\max }}+\frac{1}{5} \cdot\left(a_{1}\right)^{2} \cdot\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{5}\right]_{0}^{\tilde{z}_{\max }}+ \\
& \frac{2}{3} \cdot \tilde{v}_{\max } \cdot a_{2} \cdot\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{3}\right]_{\tilde{z}_{\max }}^{1} \\
& +\frac{1}{5} \cdot\left(a_{2}\right)^{2} \cdot\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{5}\right]_{\tilde{z}_{\max }}^{1}
\end{aligned}
$$

Upon introducing the previously-mentioned integration limits, for the Boussinesq coefficient $\beta$, results in the next algebraic expression (20):

$$
\begin{aligned}
& \beta=\left(\tilde{v}_{\max }\right)^{2}+\frac{2}{3} \cdot \tilde{v}_{\max } \cdot a_{1} \cdot\left(\tilde{z}_{\max }\right)^{3}+\frac{1}{5} \cdot\left(a_{1}\right)^{2} \cdot\left(\tilde{z}_{\max }\right)^{5}+ \\
& \frac{2}{3} \cdot \tilde{v}_{\max } \cdot a_{2} \cdot\left(1-\tilde{z}_{\max }\right)^{3}+\frac{1}{5} \cdot\left(a_{2}\right)^{2} \cdot\left(1-\tilde{z}_{\max }\right)^{5}
\end{aligned}
$$

Going over similar calculation stages for coefficient $\alpha$, successively results in:

$$
\begin{aligned}
& \alpha=\int_{0}^{1} \mathrm{f}^{3}(\tilde{z}) \cdot d \tilde{z}= \\
& \int_{0}^{\tilde{z}_{\text {max }}}\left[\tilde{v}_{\text {max }}+a_{1} \cdot\left(\tilde{z}-\tilde{z}_{\text {max }}\right)^{2}\right]^{3} \cdot \mathrm{~d} \tilde{z} \\
& +\int_{\tilde{z}_{\text {max }}}^{1}\left[\tilde{v}_{\text {max }}+a_{2} \cdot\left(\tilde{z}-\tilde{z}_{\text {max }}\right)^{2}\right]^{3} \cdot \mathrm{~d} \tilde{z} \\
& \alpha=\left(\tilde{v}_{\text {max }}\right)^{3} \int_{0}^{\tilde{z}_{\text {max }}} \mathrm{d} \tilde{z}+3 a_{1}\left(\tilde{v}_{\text {max }}\right)^{2} \int_{0}^{\tilde{z}_{\text {max }}}\left(\tilde{z}-\tilde{z}_{\text {max }}\right)^{2} \mathrm{~d} \tilde{z} \\
& +3\left(a_{1}\right)^{2} \tilde{v}_{\max } \int_{0}^{\tilde{z}_{\max }}\left(\tilde{z}-\tilde{z}_{\max }\right)^{4} \mathrm{~d} \tilde{z}+\left(a_{1}\right)^{3} \int_{0}^{\tilde{z}_{\max }}\left(\tilde{z}-\tilde{z}_{\max }\right)^{6} \mathrm{~d} \tilde{z} \\
& \left(\tilde{v}_{\text {max }}\right)^{3} \int_{\tilde{z}_{\text {max }}}^{1}+\mathrm{d} \tilde{z}+3 a_{2}\left(\tilde{v}_{\text {max }}\right)^{2} \int_{\tilde{z}_{\text {max }}}^{1}\left(\tilde{z}-\tilde{z}_{\text {max }}\right)^{2} \cdot \mathrm{~d} \tilde{z}+ \\
& 3\left(a_{2}\right)^{2} \tilde{v}_{\max } \int_{\tilde{z}_{\max }}^{1}\left(\tilde{z}-\tilde{z}_{\text {max }}\right)^{4} \cdot \mathrm{~d} \tilde{z}+\left(a_{2}\right)^{3} \int_{\tilde{z}_{\max }}^{1}\left(\tilde{z}-\tilde{z}_{\text {max }}\right)^{6} \cdot \mathrm{~d} \tilde{z} \\
& \alpha=\left(\tilde{v}_{\max }\right)^{3} \int_{0}^{1} \mathrm{~d} \tilde{z}+a_{1} \cdot\left(\tilde{v}_{\max }\right)^{2}\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{3}\right]_{0}^{\tilde{\mathrm{z}}_{\text {max }}} \\
& +\frac{3}{5}\left(a_{1}\right)^{2} \tilde{v}_{\max }\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{5}\right]_{0}^{\tilde{z}_{\max }}+\frac{1}{7}\left(a_{1}\right)^{3}\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{7}\right]_{0}^{\tilde{\mathrm{m}}_{\max }} \\
& +a_{2} \cdot\left(\tilde{v}_{\max }\right)^{2}\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{3}\right]_{\tilde{z}_{\max }}^{1}+\frac{3}{5}\left(a_{2}\right)^{2} \tilde{v}_{\max } \\
& {\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{5}\right]_{\tilde{z}_{\max }}^{1}+\frac{1}{7}\left(a_{2}\right)^{3}\left[\left(\tilde{z}-\tilde{z}_{\max }\right)^{7}\right]_{\tilde{z}_{\max }}^{1}}
\end{aligned}
$$

$$
\begin{align*}
& \alpha=\left(\tilde{v}_{\max }\right)^{3}+a_{1}\left(\tilde{v}_{\max }\right)^{2}\left[\left(\tilde{z}_{\max }\right)^{3}\right]+\frac{3}{5}\left(a_{1}\right)^{2} \\
& \tilde{v}_{\max }\left[\left(\tilde{z}_{\max }\right)^{5}\right]+\frac{1}{7}\left(a_{1}\right)^{3}\left[\left(\tilde{z}_{\max }\right)^{7}\right]+  \tag{21}\\
& +a_{2}\left(\tilde{v}_{\max }\right)^{2}\left[\left(1-\tilde{z}_{\max }\right)^{3}\right]+\frac{3}{5}\left(a_{2}\right)^{2} \tilde{v}_{\max } \\
& {\left[\left(1-\tilde{z}_{\max }\right)^{5}\right]+\frac{1}{7}\left(a_{2}\right)^{3}\left[\left(1-\tilde{z}_{\max }\right)^{7}\right]}
\end{align*}
$$

Introducing into equations (20) and (21) expressions (10) for coefficients $a_{1}$ and $a_{2}$, after performing the elementary calculations, ultimately results in the following expressions for coefficients $\beta$ and $\alpha$, depending only on the quota (level) and the characteristic relative velocities:

$$
\beta=\left(\tilde{v}_{\max }\right)^{2}-\frac{1}{5}\left[\begin{array}{l}
\left(\frac{7}{3} \cdot \tilde{v}_{\max }+\tilde{v}_{f}\right)\left(\tilde{v}_{\max }-\tilde{v}_{f}\right)  \tag{22}\\
\left(\tilde{z}_{\max }\right)+\left(\frac{7}{3} \cdot \tilde{v}_{\max }+\tilde{v}_{s}\right) \\
\left(\tilde{v}_{\max }-\tilde{v}_{s}\right)\left(1-\tilde{z}_{\max }\right)
\end{array}\right]
$$

and:

$$
\begin{align*}
& \alpha=\left(\tilde{v}_{\max }\right)^{3}-\left(\tilde{v}_{\max }-\tilde{v}_{f}\right) \cdot \tilde{z}_{\max } \\
& {\left[\begin{array}{l}
\left(\tilde{v}_{\max }\right)^{2}-\frac{3}{5}\left(\tilde{v}_{\max }-\tilde{v}_{f}\right) \cdot \tilde{v}_{\max } \\
+\frac{1}{7}\left(\tilde{v}_{\max }-\tilde{v}_{f}\right)^{2}
\end{array}\right]+} \\
& +\left(\tilde{v}_{\max }-\tilde{v}_{s}\right) \cdot\left(1-\tilde{z}_{\max }\right) \cdot  \tag{23}\\
& \cdot\left[\begin{array}{l}
\left(\tilde{v}_{\max }\right)^{2}-\frac{3}{5}\left(\tilde{v}_{\max }-\tilde{v}_{s}\right) \cdot \tilde{v}_{\max } \\
+\frac{1}{7}\left(\tilde{v}_{\max }-\tilde{v}_{s}\right)^{2}
\end{array}\right]
\end{align*}
$$

With notations:

$$
\begin{equation*}
\Delta \tilde{v}_{\max _{-} f}=\tilde{v}_{\max }-\tilde{v}_{f}, \Delta \tilde{v}_{\max _{-} s}=\tilde{v}_{\max }-\tilde{v}_{s} \tag{24}
\end{equation*}
$$

relations (22) and (23) were finally rewritten in as follows:

$$
\beta=\left(\tilde{v}_{\max }\right)^{2}-\frac{1}{5}\left[\begin{array}{l}
\Delta \tilde{v}_{\max _{-} f} \cdot\left(\frac{7}{3} \cdot \tilde{v}_{\max }+\tilde{v}_{f}\right) \cdot \tilde{z}_{\max }  \tag{25}\\
+\Delta \tilde{v}_{\max \_s} \cdot\left(\frac{7}{3} \tilde{v}_{\max }+\tilde{v}_{s}\right) \cdot\left(1-\tilde{z}_{\max }\right)
\end{array}\right]
$$

$$
\begin{align*}
& \alpha=\left(\tilde{v}_{\max }\right)^{3}-\Delta \tilde{v}_{\max _{-} f} \cdot \tilde{z}_{\max } \cdot\left[\begin{array}{l}
\left(\tilde{v}_{\max }\right)^{2}-\frac{3}{5} \cdot \\
\Delta \tilde{v}_{\max \_f} \cdot \tilde{v}_{\max } \\
+\frac{1}{7}\left(\Delta \tilde{v}_{\max _{-} f}\right)^{2}
\end{array}\right]+ \\
& -\Delta \tilde{v}_{\max _{-} s} \cdot\left(1-\tilde{z}_{\max }\right) \cdot\left[\begin{array}{l}
\left(\tilde{v}_{\max }\right)^{2}-\frac{3}{5} \cdot \Delta \tilde{v}_{\max _{-} s} \\
\cdot \tilde{v}_{\max }+\frac{1}{7}\left(\Delta \tilde{v}_{\max _{-} s}\right)^{2}
\end{array}\right] \tag{26}
\end{align*}
$$

## CASE STUDY

The basic data referring to the case study regard a trapezoidal channel made of concrete from the Irrigation and Drainage Laboratory of the Technical University in Iasi, with a bottom width $b$ $=0.60 \mathrm{~m}$, angular coefficient $m=0.89$ and roughness coefficient $n=0.014$. At a water depth of $h=0.49 \mathrm{~m}$ and hydraulic slope of $I_{h}=0.175 \%$, the channel transported a flow of $Q=0.198 \mathrm{~m}^{3} \mathrm{~s}^{-1}$, with the average velocity of the entire section being $0.39 \mathrm{~m} \mathrm{~s}^{-1}$.

The vertical taken into consideration in performing the velocity profile passes through the channel's axis, the hydraulic radius corresponding to this vertical being $R=h=0.49 \mathrm{~m}$. Applying formulae (2), (3) and (6) resulted in:

$$
\begin{aligned}
& v_{f} \cong v_{*}=\sqrt{g \cdot h \cdot I_{h}}=\sqrt{9.81 \cdot 0.49 \cdot 0.000175}= \\
& =0.0290 \mathrm{~ms}^{-1} \\
& V=C \sqrt{R \cdot I_{h}} \cong \frac{1}{n} h^{1 / 6} \sqrt{h \cdot I_{h}}= \\
& =\frac{1}{0.014} \cdot 0.49^{1 / 6} \sqrt{0.49 \cdot 0.000175}= \\
& =0.5873 \mathrm{~ms}^{-1}>0.39 \mathrm{~ms}^{-1}
\end{aligned}
$$

$\tilde{v}_{f} \cong \frac{v_{*}}{V}=\frac{0.029}{0.5873}=0.049385 ; \tilde{v}_{f} \cong n \sqrt{g} \cdot h^{-1 / 6}=$ $=0.014 \cdot \sqrt{9.81} \cdot 0.49^{-1 / 6}=0.049385$

Making use of relations (1), for the middle of the variation interval for $\tilde{z}_{\text {max }}(5)$, with $\varphi \cong 1.29$, as well as (16), respectively resulted in:

$$
\tilde{z}_{\max }=z_{\max } / h=(4 / 5+5 / 6) / 2=0.8167
$$

$$
\begin{aligned}
& \tilde{v}_{s}=\frac{3-\tilde{v}_{f} \cdot \tilde{z}_{\max }}{1+2 \cdot \varphi-\tilde{z}_{\max }}=\frac{3-0.049385 \cdot 0.8167}{1+2 \cdot 1.29-0.8167}= \\
& =1,07106, \tilde{v}_{\max }=\varphi \cdot \tilde{v}_{s}=1.29 \cdot 1.07106=1.38167
\end{aligned}
$$

From relations (10), coefficients $a_{1}$ and $a_{2}$ were calculated, and in addition the real expression of the function was established (11):

$$
\begin{aligned}
& a_{1}=-\frac{\tilde{v}_{\max }-\tilde{v}_{f}}{\left(\tilde{z}_{\max }\right)^{2}}=-\frac{1.38167-0.049385}{(0.8167)^{2}}= \\
& =-1.9974336, \\
& a_{2}=-\frac{\tilde{v}_{\max }-\tilde{v}_{s}}{\left(1-\tilde{z}_{\max }\right)^{2}}=-\frac{1.38167-1.07106}{(1-0.8167)^{2}}= \\
& =-9.244591
\end{aligned}
$$

$$
\tilde{v}=\mathrm{f}(\tilde{z})=\left\{\begin{array}{l}
1.38167-1.9974336 \cdot(\tilde{z}-0.8167)^{2}  \tag{27}\\
\text { for } 0 \leq \tilde{z} \leq 0.8167 \\
1.38167-9.244591 \cdot(\tilde{z}-0.8167)^{2} \\
\text { for } 0.8167 \leq \tilde{z} \leq 1
\end{array}\right.
$$

Relation (15) and relation (12) are verified analytically and numerically, respectively, resulting in:

$$
\begin{aligned}
& {\left[2 \cdot \tilde{v}_{\max }+\tilde{v}_{s}-\left(\tilde{v}_{s}-\tilde{v}_{f}\right) \cdot \tilde{z}_{\max }\right] / 3=} \\
& =\left[\begin{array}{l}
2 \cdot 1.38167+1.07106- \\
(1.07106-0.049385) \cdot 0.8167
\end{array}\right] / 3= \\
& =1.0000001924666 \\
& \int_{0}^{1} \mathrm{f}(\tilde{z}) \cdot \mathrm{d} \tilde{z}=1.0000001924665
\end{aligned}
$$

Hence the analytical expression (27) for the velocity profile in relative coordinates, on the considered vertical, is correct and accurate.

Following, the the above-presented applications - the values for the relative quota $\tilde{z}_{\text {med }}$ and for the coefficients $\beta$ and $\alpha$ - were determinated; thus using relations (17), (24), (25) and (26) respectively resulted in:

$$
\begin{aligned}
& \tilde{z}_{\text {med }}=\frac{1-\tilde{v}_{f}}{\tilde{v}_{\max }-\tilde{v}_{f}} \cdot \tilde{z}_{\max }= \\
& =\frac{1-0.049385}{1.38167-0.049385} \cdot 0.8167= \\
& =0.379572 \\
& \Delta \tilde{v}_{\text {max } \_f}=\tilde{v}_{\max }-\tilde{v}_{f}=1.38167-0.049385= \\
& =1.3322855, \\
& \Delta \tilde{v}_{\text {max } \_s}=\tilde{v}_{\text {max }}-\tilde{v}_{s}=1.38167-1.07106= \\
& =0.310608 \\
& \beta=1.38167^{2}-\frac{1}{5}\left[\left(\frac{7}{3} \cdot 1.38167+0.049385\right)\right]- \\
& -\frac{1}{5}\left[\left(\frac{7}{3} .1 .38167+1.07106\right) \cdot 0.310608\right]= \\
& =1.14779085 \\
& \alpha=1.38167^{3}-0.8167 \cdot 1.3322855 .
\end{aligned}
$$

$$
\cdot\left(1.38167^{2}-\frac{3}{5} \cdot 1.3322855 \cdot 1.38167+\frac{1}{7} \cdot 1.3322855^{2}\right)-
$$

$$
0.310608 \cdot(1-0.8167)\binom{1.38167^{2}-\frac{3}{5} \cdot 1.3322855 \cdot}{\cdot 1.38167+\frac{1}{7} \cdot 1.3322855^{2}}=
$$

$$
=1.39150495
$$

The values for coefficients $\beta$ and $\alpha$ were obtained and by the numeric integration of relations (18) and (19), using the Lobato method, and presented in MATLAB with the standard external function quadl.m, resulting in values compatible with those provided by relations (25) and (26):
$\beta=\int_{0}^{1} \mathrm{f}^{2}(\tilde{z}) \cdot d \tilde{z}=1.147790849065254$,
$\alpha=\int_{0}^{1} \mathrm{f}^{3}(\tilde{z}) \cdot d \tilde{z}=1.391504948866777$
The graphical representation of function (27) lead to the vertical velocity profile presented in Figure 1; imposed points are marked (8) on the graphic representation, as well as the point corresponding to the average velocity $\left(V, \tilde{z}_{\text {med }}\right)$.

The graphical representation of function (27) lead to the vertical velocity profile presented in

Figure 1; imposed points are marked (8) on the graphic representation, as well as the point corresponding to the average velocity $\left(V, \tilde{z}_{\text {med }}\right)$.

If the hydraulic slope is also indicated $I_{h}$ (or if it can be approximated with the channel's slope
thalweg), then the vertical average velocity can also be estimated, and in the end the velocity profile drafted following a vertical axis and in absolute coordinates $(z, v)$; obviously the aspect of this draft is identical to the one in figure 1.


Figure 1. Relative velocity profile according to the vertical axis drawn in the channel's axis.

## CONCLUSIONS

The analytical relation for the velocity profile according to a vertical axis, in relative coordinates, was established using minimum channel data: depth $h$, roughness $n$, relative quota (level) at which the maximum velocity is recorded $\tilde{z}_{\text {max }}$ and the relation between the velocity at the surface stream and the maximum velocity, $\varphi$.

If, in addition, the hydraulic slope is also indicated (which can be estimated with the channel's thalweg slope) the velocity profile can be estimated according to a vertical axis and in absolute coordinates $(z, v)$.

Analytical relations were deduced for the relative velocities at the bottom, maximum and at the water surface, as well as for the relative quota corresponding to the average velocity.

Analytical relations for Coriolis and Boussinesq coefficients were deduced, relations were verified using a performant numerical integration method.

Systematic experimental research is necessary for accurately establishing the relative quota $\tilde{z}_{\text {max }}$, coefficient $\varphi$ and for the analysis of the accuracy of the proposed double parabolic distribution law.

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